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Ternary universal sums of generalized polygonal numbers

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서방남

Ternary universal sums of generalized polygonal numbers

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Abstract

An integer of the form $p_m(x) = \frac{(m-2)x^2-(m-4)x}{2}$ $(m \geq 3)$, for some integer x is called a generalized polygonal number of order m. A ternary sum $\Phi^{a,b,c}_{i,j,k}(x,y,z) = ap_i(x) + bp_j(y) + cp_k(z)$ of generalized polygonal numbers, for some positive integers a, b, c and some integers $i, j, k \geq 3$, is said to be *universal* if for any positive integer n, the equation $\Phi^{a,b,c}_{i,j,k}(x,y,z) = n$ has an integer solution x, y, z. In this article, we prove the universalities of 15 ternary sums of generalized polygonal numbers listed in (1.0.2). This was conjectured by Z.-W. Sun.

Key words: Generalized polygonal number, Universal sums of polygonal number, Ternary sums of polygonal number, Representation of ternary quadratic form

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Contents

Abstract **i**

1 Introduction **1**

2 Preliminaries **5**

2.1 Definitions 5

2.2 Representations of quadratic spaces and lattices 9

2.3 Representation theory of ternary form 11

3 Universal sums of polygonal numbers **14**

3.1 Sums of polygonal numbers and representation of ternary forms. 14

4 Ternary universal sums of polygonal numbers **22**

Abstract (in Korean) **41**

Acknowledgement (in Korean) **42**

Chapter 1

Introduction

Let m be any positive integer greater than two. An integer of the form

$$p_m(x) = \frac{(m-2)x^2 - (m-4)x}{2}$$

for some nonnegative integer x is said to be a *polygonal number of order m* (or m-gonal number). Hence, a polygonal number is a number represented as dots or pebbles arranged in the shape of a regular polygon. If the variable x is an integer, $p_m(x)$ is called a *generalized polygonal number of order m* (or generalized m-gonal number). For example, the following integers $0,\ 1,\ 3,\ 6,\ 10,\ 15, \cdots$ are triangular numbers and $0,\ 1,\ 4,\ 9,\ 25,\ 36, \cdots$ are squares, and $0,\ 1,\ 2,\ 5,\ 7,\ \ 12,\ 15, \cdots$ are generalized pentagonal numbers and so on. By definition, every polygonal number of order m is a generalized polygonal number of order m. However, the converse is true only when $m = 3, 4$. If m is greater than 4, the set of all m-gonal numbers is a proper subset of all generalized m-gonal numbers.

A famous assertion of Fermat states that each positive integer can be expressed as a sum of three triangular numbers, equivalently every positive integer that is congruent to 3 modulo 8 is a sum of three (odd) squares. This follows immediately from the well known theorem of Gauss and Legendre, which states that every positive integer is a sum of three squares if and only if it is not of the form $4^k(8l+7)$ for some non negative integers k and l.

In general, a ternary sum $\Phi_{i,j,k}^{a,b,c} = ap_i(x) + bp_j(y) + cp_k(z)$ $(a, b, c > 0)$ of polygonal numbers is called *universal over* $\mathbb{N}$ if for any non negative integer n, the diophantine equation

$$ap_i(x) + bp_j(y) + cp_k(z) = n \tag{1.0.1}$$

has a *non negative* integer solution x, y, z. It is called *universal over* $\mathbb{Z}$ if the above equation has an integer solution.

In 1862, Lioville determined all ternary universal sums of triangular numbers. More precisely, for positive integers a, b, c ($a \le b \le c$), he proved that the ternary sum $ap_3(x) + bp_3(y) + cp_3(z)$ of triangular numbers is universal if and only if (a, b, c) is one of the following triples:

$$(a, b, c) = (1, 1, 1),\ (1, 1, 2),\ (1, 1, 4), (1, 1, 5),\ (1, 2, 2),\ (1, 2, 3),\ (1, 2, 4).$$

In 2007, Zhi Wei Sun suggested the problem on determination of all ternary universal sums of the form $ap_3(x) + bp_3(y) + cp_4(z)$ or $ap_3(x) + bp_4(y) + cp_4(z)$, and this was completed by Sun and his collaborates (see [1], [7] and [8]). In fact, the ternary sum $ap_3(x) + bp_3(y) + cp_4(z)$ is universal if and only if (a, b, c) is one of the following triples:

$$\begin{aligned}(a, b, c) = \ & (1, 1, 1),\ (1, 1, 2),\ (1, 1, 4),\ (1, 2, 1),\ (1, 2, 2),\\ & (1, 2, 3),\ (1, 2, 4),\ (2, 2, 1),\ (2, 4, 1),\ (2, 5, 1),\\ & (1, 3, 1),\ (1, 4, 1),\ (1, 4, 2),\ (1, 6, 1),\ (1, 8, 1),\end{aligned}$$

and the ternary sum $ap_3(x)+bp_4(y)+cp_4(z)$ is universal if and only if (a, b, c) is one of the following triples:

$$\begin{aligned}(a, b, c) = \ & (1, 1, 1),\ (1, 1, 2),\ (1, 1, 3),\ (1, 1, 4),\ (1, 1, 8),\\ & (1, 2, 2),\ (2, 1, 1),\ (2, 1, 2),\ (2, 1, 4),\ (4, 1, 2).\end{aligned}$$

Note that if the above sums are universal over $\mathbb{Z}$, then they are universal over $\mathbb{N}$, for any generalized triangular (square) number is a triangular (square, respectively) number.

Sun gave in [9] a complete list of candidates of all possible ternary universal sums of (generalized) polygonal numbers. In particular, he proved that there are at most twenty ternary universal sums over $\mathbb{Z}$ which are of the form $ap_k(x) + bp_k(y) + cp_k(z)$ for $k = 5$ or $k \geq 7$, and conjectured that these are all universal over $\mathbb{Z}$. They are, in fact, $k = 5$ and

$$\begin{aligned}(a, b, c) = \quad & (1, 1, s) \quad 1 \leq s \leq 10, \quad s \neq 7, \\ & (1, 2, 2), \quad (1, 2, 3), \quad (1, 2, 4), \quad (1, 2, 6), \quad (1, 2, 8), \\ & (1, 3, 3), \quad (1, 3, 4), \quad (1, 3, 6), \quad (1, 3, 7), \quad (1, 3, 8), \quad (1, 3, 9).\end{aligned}$$

Guy proved in [3] the universality for the case when $k = 5$ and $(a, b, c) = (1, 1, 1)$ (for the complete proof, see [9]). In the same article [9] as above, Sun also proved the universality for the cases when $k = 5$

$$(a, b, c) = (1, 1, 2), \quad (1, 1, 4), \quad (1, 2, 2), \quad (1, 2, 4), \quad (1, 1, 5), \quad (1, 3, 6).$$

Note that the set of generalized hexagonal numbers equals to the set of triangular numbers. Shortly after publishing this result Ge and Sun proved in [2] the universality for the cases when $k = 5$ and

$$(a, b, c) = (1, 1, 3), \quad (1, 2, 3), \quad (1, 2, 6), \quad (1, 3, 3), \quad (1, 3, 4), \quad (1, 3, 9).$$

The universality of the remaining seven candidates of ternary universal sums of pentagonal numbers was proved by Oh in [6].

Recently, Sun gave in [9] a complete list of 95 candidates of ternary sums of polygonal numbers that are universal over $\mathbb{N}$ under this assumption.

In fact, the proof of the universality over $\mathbb{Z}$ of a ternary sum of polygonal numbers is much easier than the proof of universality over $\mathbb{N}$. For this reason, one may naturally ask the universalities over $\mathbb{Z}$ of 95 candidates of ternary sums. Related with this question, Sun and Ge proved in [2] and [9] that 62 ternary sums among 95 candidates are, in fact, universal over $\mathbb{Z}$. The remaining $33 = 95 - 62$ candidates are as follows:

$$\begin{array}{llll}
\mathbf{p_3+9p_3+p_5}, & \mathbf{p_3+2p_4+4p_5}, & \mathbf{p_3+4p_4+2p_5}, & \mathbf{p_3+2p_3+p_7}, \\
\mathbf{p_3+p_4+2p_7}, & \mathbf{p_3+p_5+p_7}, & p_3+p_5+4p_7, & \mathbf{p_3+2p_5+p_7}, \\
\mathbf{p_3+p_7+2p_7}, & \mathbf{p_3+2p_3+2p_8}, & \mathbf{p_3+p_7+p_8}, & \mathbf{p_3+2p_3+p_9}, \\
p_3+2p_3+2p_9, & \mathbf{p_3+p_4+p_9}, & p_3+p_4+2p_9, & p_3+2p_4+p_9, \\
p_3+p_5+2p_9, & \mathbf{2p_3+p_5+p_9}, & \mathbf{p_3+p_3+p_{12}}, & \mathbf{p_3+2p_3+p_{12}}, \\
p_3+2p_3+2p_{12}, & p_3+p_4+p_{13}, & p_3+p_5+p_{13}, & p_3+2p_3+p_{15}, \\
p_3+p_4+p_{15}, & p_3+2p_3+p_{16}, & p_3+p_3+p_{17}, & p_3+2p_3+p_{17}, \\
p_3+p_4+p_{17}, & p_3+2p_4+p_{17}, & p_3+p_4+p_{18}, & p_3+2p_3+p_{23}, \\
p_3+p_4+p_{27}. & & &
\end{array} \tag{1.0.2}$$

In this thesis, we prove that 15 ternary sums written in boldface in the above remaining candidates are universal over $\mathbb{Z}$. Most results given in this thesis are done by joint work with B.-K. Oh.

Main idea of our method is to reduce ternary sums of generalized polygonal numbers to the representations of ternary quadratic forms without any congruence condition. In most cases, we use some modified version of the method developed in [6].

In Chapter 2, we introduce notations and terminologies that will be used in this thesis. Moreover, for those who are unfamiliar with it, we review the method developed in [6] and introduce the main theorem.

In Chapter 3, We introduce our main problem. We will show that in many cases, representations of integers by ternary sums of generalized polygonal numbers can be reduced to the representations of integers by ternary quadratic forms.

Finally, in Chapter 4, we prove that 15 ternary sums of generalized polygonal numbers given above are universal over $\mathbb{Z}$.

Chapter 2

Preliminaries

In this chapter, we will introduce from basic concepts, some definitions and properties on the theory of quadratic forms.

2.1 Definitions

Let $\mathbb{Q}$ and $\mathbb{R}$ be the field of rational numbers and the field of real numbers, respectively. Let S be the set of all primes including ∞. For a prime $p \in S$, we write the fields of p-adic completions of $\mathbb{Q}$ by $\mathbb{Q}_p$ and $\mathbb{Q}_\infty = \mathbb{R}$. For a finite prime p, $\mathbb{Z}_p$ denotes the p-adic integer ring.

Let R be the ring of integers $\mathbb{Z}$ or the ring of p-adic integers $\mathbb{Z}_p$. Let L be an *R-lattice* by which we mean a free R-module of finite rank equipped with a non-degenerate symmetric bilinear form $B : L \times L \to R$ with following properties:

$$B(x, y + z) = B(x, y) + B(x, z)$$
$$B(ax, y) = \alpha B(x, y),\ B(x, y) = B(y, x)$$

for all $x, y, z \in L$ and all $\alpha \in R$. If we put $Q(x) = B(x, x)$ for all $x \in L$, we get the quadratic map $Q : L \to R$. The following identities hold:

$$Q(\alpha x) = \alpha^2 Q(x)$$
$$Q(x + y) = Q(x) + Q(y) + 2B(x, y)$$

for all $x, y \in L$ and $\alpha \in R$.
Let $e_1, e_2, \ldots, e_n$ be a basis of L. The corresponding symmetric matrix M_L to (L, B) is defined by

$$M_L = (B(e_i, e_j)).$$

We abuse this symmetric matrix M_L with the lattice L. The *determinant* $det(M_L)$ of the symmetric matrix M_L is called the *discriminant* dL of L. Note that the discriminant dL is uniquely determined up to unit squares of R.
We define *scale* $\mathfrak{s}(L)$ of L to be the ideal of R generated by $B(x, y)$ for any $x, y \in L$, *norm* $\mathfrak{n}(L)$ of L to be the ideal of R generated by $Q(x)$ for any $x \in L$. For a non-zero $\alpha \in R$, we denote by $L^{(\alpha)}$ the R-lattice obtained from scaling L by α.
Suppose that L and M are R-lattices. Then a linear map $\sigma : L \longrightarrow M$ is *representation* of L into M if

$$B(x, y) = B(\sigma(x), \sigma(y))$$

for any $x, y \in L$. Moreover, if L is represented by M, we write $L \to M$.
For two R-lattices L and M, we say that M is *isometric* to L if $L \to M$ and $M \to L$. In this case, we write $M \simeq L$.
For two quadratic spaces W and V, we define $W \to V$ and $W \simeq V$ similarly as above.
Let L be a $\mathbb{Z}$-lattice. Then lattice L is said to be *positive definite (or indefinite)* if symmetric matrix M_L is positive definite (or indefinite, respectively). We say that L is even if $Q(x) \in 2\mathbb{Z}$ for all $x \in L$. We say that L is odd otherwise.
We define $L_{\mathbb{Q}} := L \otimes \mathbb{Q}$ to be a quadratic space over $\mathbb{Q}$, and $L_p := L \otimes \mathbb{Z}_p$ to be a $\mathbb{Z}_p$-lattice.
We say that a $\mathbb{Z}$-lattice L is *unimodular* if $dL = \pm 1$.
We denote by I_n the $\mathbb{Z}$-lattice of rank n whose corresponding symmetric matrix is the identity matrix.
For a sublattice l of a $\mathbb{Z}$-lattice L, we say that l is a *primitive sublattice* of L

if $l_{\mathbb{Q}} \cap L = l$. For sublattices L_1 and L_2 of L, we write $L = L_1 \perp L_2$ if

$$L = L_1 \oplus L_2 \ \text{ and } \ B(x, y) = 0 \ \text{ for all } x \in L_1,\ y \in L_2.$$

For a $\mathbb{Z}$-lattice L, we denote by L^m the direct sum of m-copies of L. We say that L is *decomposable* if $L \simeq L_1 \perp L_2$ for some non-zero lattices L_1, L_2. L is said to be *indecomposable* otherwise.
We define the *dual lattice* $L^\sharp$ of L by

$$L^\sharp := \{x \in L \otimes \mathbb{Q} \mid B(x, L) \subseteq \mathbb{Z}\}.$$

Note that $L \subseteq L^\sharp$ and $|L^\sharp/L| = |dL|$. For a sublattice l of L, we define

$$l^\perp := \{x \in L \mid B(x, L) = 0\},$$

which is also a sublattice of L. Note that $dl \cdot dl^\perp = t^2 dL$ for some integer t and $dl^\perp$ divides $dl \cdot dL$ (see [[13], 5.3.3]).
For a sublattice l of $L \perp M$ of the form

$$l = \mathbb{Z}(x_1 + y_1) + \mathbb{Z}(x_2 + y_2) + \cdots + \mathbb{Z}(x_n + y_n)$$

with $x_i \in L$, $y_i \in M$, we define

$$l(L) := \mathbb{Z}x_1 + \mathbb{Z}x_2 + \cdots + \mathbb{Z}x_n \quad \text{ and } \quad l(M) := \mathbb{Z}y_1 + \mathbb{Z}y_2 + \cdots + \mathbb{Z}y_n.$$

Even when $\phi : l \to L \perp M$, we simply write $l(L)$ instead of $\phi(l)(L)$ if no confusion may arise.
A lattice l is said to be *additively indecomposable* if either $l(L) = 0$ or $l(M) = 0$ whenever $l \to L \perp M$. It is well known that for each rank n, there exist only finitely many positive definite additively indecomposable $\mathbb{Z}$-lattices (see [14]).
Note that every additively indecomposable $\mathbb{Z}$-lattice is indecomposable, but the converse is not true in general. However, in the unimodular case, the converse is also true (see, for example, [14]).

A $\mathbb{Z}$-lattice L is called odd if $\mathfrak{n}(L) = \mathbb{Z}$, even otherwise.

The term "lattice" will always refer to a positive definite integral $\mathbb{Z}$-lattice on an n-dimensional positive definite quadratic space over $\mathbb{Q}$. Let $L = \mathbb{Z}x_1 + \cdots + \mathbb{Z}x_n$ be a $\mathbb{Z}$-lattice of rank n. We write

$$L \simeq (B(x_i, x_j)).$$

The right hand side matrix is called a *matrix presentation* of L. If $(B(x_i, x_j))$ is diagonal, then we simply write $L \simeq \langle Q(x_1), \cdots, Q(x_n)\rangle$.
We define the genus of a $\mathbb{Z}$-lattice L to be the set of all $\mathbb{Z}$-lattices M with the following property : For each prime p, M_p is isometric to L_p and also $M_{\mathbb{Q}}$ is isometric to $L_{\mathbb{Q}}$, i.e, the set of all lattices that are locally isometric to L, which is denoted by $\text{gen}(L)$. The subset of $\text{gen}(L)$ consisting of all lattices that are isometric to L itself is denoted by $[L]$.
For a $\mathbb{Z}$-lattice L, we define the *class number* $h(L)$ as the number of non-isometric classes in $gen(L)$. It is well known that the number of isometric classes in a genus is always finite (see, [[10], §103]).
We write $l \to gen(L)$ if l is represented by a $\mathbb{Z}$-lattice M in $gen(L)$.
An integer a is called *an eligible integer of* L if it is locally represented by L. Note that any eligible integer of L is represented a lattice in the genus of L.The set of all eligible integers of L is denoted by $Q(\text{gen}(L))$. Similarly, $Q(L)$ denotes the set of all integers that are represented by L itself.
For an integer a, $R(a, L)$ denotes the set of all vectors $x \in L$ such that $Q(x) = a$, and $r(a, L)$ denotes its cardinality.
For a set S, we define $\pm S = \{s : s \in S \text{ or } -s \in S\}$. For two vectors $v = (v_1, v_2, \ldots, v_n), w = (w_1, w_2, \ldots, w_n) \in \mathbb{Z}^n$ and a positive integer s, if $v_i \equiv w_i \pmod{s}$ for any $i = 1, 2, \ldots, n$, then we write $v \equiv w \pmod{s}$. Any unexplained notations and terminologies can be found in [13] or [10].

2.2 Representations of quadratic spaces and lattices

We begin with well-known theorems concerning representations of quadratic forms over $\mathbb{Q}$ and over $\mathbb{Z}_p$. The following two theorems are one of the most important facts and give a complete answer for the representations of quadratic forms over $\mathbb{Q}$.

Theorem 2.2.1. *Let U and V be quadratic spaces over $\mathbb{Q}_p$ with $\nu = dimV - dimU \geq 0$. Then U is represented by V if and only if*

$$\begin{array}{ll} U \simeq V & \text{when } \nu = 0 \\ U \perp \langle dU \cdot dV \rangle \simeq V & \text{when } \nu = 1 \\ U \perp H \simeq V & \text{when } \nu = 2, \quad dU = -dV. \end{array}$$

Here, H denotes the hyperbolic plane.

Proof. See [[10], 63:21]. □

Theorem 2.2.2. *(Hasse-Minkowski Theorem) Let U and V are quadratic space over $\mathbb{Q}$. Then U is represented by V if and only if $U_p = U \otimes \mathbb{Q}_p$ is represented by $V_p = V \otimes \mathbb{Q}_p$ for all $p \in S$ (including ∞).*

Proof. See [[10], 66:3]. □

Suppose L and M be given $\mathbb{Z}$-lattices. If M is represented by L then M_p is represented by L_p for all prime p. In general, the converse is not true.
Let L be a $\mathbb{Z}_p$-lattice of rank n. If $(dL)\mathbb{Z}_p$ is equal to $\mathfrak{s}(L)^n$, then we call L a *modular lattice.* If a modular lattice L over $\mathbb{Z}_p$ is split by a sublattice of rank 1, L is called *proper.* Otherwise, we say that L is *improper.* Note that for odd prime, p every modular lattice over $\mathbb{Z}_p$ is proper.
If we group the modular components of splitting $1-$ and $2-$dimensional modular lattices in a suitable way we find L has a splitting $L = L_1 \perp L_2 \perp \ldots \perp L_t$ in which each component is modular and $\mathfrak{s}(L_t) \subsetneqq \mathfrak{s}(L_{t-1}) \cdots \subsetneqq \mathfrak{s}(L_1)$. We call such a splitting a *Jordan decomposition* for L or *Jordan splitting of L*

.See [[10], §91.C].

Definition 2.2.3. For given $\mathbb{Z}_p$-lattices l and L, let $l =\perp_{\lambda=1}^{m} l_\lambda$ and $L =\perp_{\lambda=1}^{t} L_\lambda$ be their Jordan decompositions. We define $\mathfrak{l}_i :=\perp l_\mu$, where μ runs over the indices for which $\mathfrak{s}(l_\mu) \supseteq p^i\mathbb{Z}_p$. We define $\mathfrak{L}_i :=\perp L_\mu$ for L similarly.

Theorem 2.2.4. *For odd prime p, l and L be $\mathbb{Z}_p$-lattices and let $l =\perp l_\lambda$ and $L =\perp L_\lambda$ be their Jordan decompositions. Then $l \to L$ if and only if $\mathfrak{l}_i \otimes \mathbb{Q}_p \to \mathfrak{L}_i \otimes \mathbb{Q}_p$ for any i.*

Proof. See [[11], Theorem 1]. $\square$

For dyadic case, we need more notations. Let $l =\perp l_\lambda$ and $L =\perp L_\lambda$ be Jordan decompositions of l and L, respectively. We now put $\mathfrak{L}_{(i)} :=\perp L_\mu$, where μ runs over the indices for which $\mathfrak{n}(L_\mu) \supseteq 2^i\mathbb{Z}_2$, and $\mathfrak{l}_{[i]} :=\perp l_\mu$, where μ runs over the indices for which $\mathfrak{s}(l_\mu) \supseteq 2^i\mathbb{Z}_2$ and in addition, over the indices for which $\mathfrak{s}(l_\mu) = 2^{i+1}\mathbb{Z}_2$ with l_μ improper, if any.

We define Δ_i for L in the following way: If L has a proper 2^{i+1}-modular component, put $\Delta_i := 2^{i+1}\mathbb{Z}_2$; failing this, $\Delta_i := 2^{i+2}\mathbb{Z}_2$ if L has a proper 2^{i+2}-modular component; otherwise, $\Delta_i := 0$. We define δ_i for l similarly. Let $D_i := d(\mathfrak{L}_i)\mathbb{Z}_2$ and $d_i := d(\mathfrak{l}_i)\mathbb{Z}_2$; we put $D_i = 0$ ($d_i = 0$), if $\mathfrak{L}_i = 0$ ($\mathfrak{l}_i = 0$, respectively). Note that above definitions are all independent of Jordan decompositions that define them. For any fractional ideal $\mathfrak{A} \subseteq \mathbb{Q}_2$, we write $\mathfrak{A} \to U$ if there is an $x \in U$ such that $Q(x)\mathbb{Z}_2 = \mathfrak{A}$.

Definition 2.2.5. We say that l has a lower type than L if the followings hold for all i:

(1) $\dim \mathfrak{l}_i \leq \dim \mathfrak{L}_i$,

(2) $d_iD_i \to 1$ if $\dim \mathfrak{l}_i = \dim \mathfrak{L}_i$,

(3) $\delta_i \subseteq \Delta_i + 2^{i+2}\mathbb{Z}_2$ and $\Delta_{i-1} \subseteq \delta_{i-1} + 2^{i+1}\mathbb{Z}_2$ if $\dim \mathfrak{l}_i = \dim \mathfrak{L}_i$,

(4) $\Delta_{i-1} \subseteq \delta_{i-1} + 2^{i+1}\mathbb{Z}_2$ if $\dim \mathfrak{L}_i - 1 = \dim \mathfrak{l}_i > 0$ and $d_iD_i \to 2^{i+1}$,

(5) $\delta_i \subseteq \Delta_i + 2^{i+2}\mathbb{Z}_2$ if $\dim \mathfrak{L}_i - 1 = \dim \mathfrak{l}_i > 0$ and $d_iD_i \to 2^i$.

Suppose l and L be given $\mathbb{Z}_2$-lattices. If l is represented by L, we write by L/l the quadratic space satisfying $l \otimes \mathbb{Q}_2 \perp L/l \simeq L \otimes \mathbb{Q}_2$. We write $\overline{\alpha} \to U$ if either $\alpha \to U$ or $5\alpha \to U$, where $\alpha \in \mathbb{Q}_2$.

Theorem 2.2.6. *Let l have a lower type than L. Then $l \to L$ if and only if the following conditions hold for all i:*

$$\begin{aligned}
&\Delta_i {\to} \mathfrak{L}_{(i+2)}/\mathfrak{l}_{[i]},\\
&\delta_i {\to} \mathfrak{L}_{(i+2)}/\mathfrak{l}_{[i]},\\
&\mathfrak{L}_{(i+2)}/\mathfrak{l}_{[i]} \simeq \mathbb{H} \quad \textit{implies} \quad \Delta_i\delta_i \subseteq \delta_i^2,\\
&\overline{2^i} {\to} (2^i \perp \mathfrak{L}_{(i+1)})/\mathfrak{l}_i,\\
&\overline{2^i} {\to} (2^i \perp \mathfrak{L}_{i+1})/\mathfrak{l}_{[i]}.
\end{aligned}$$

Here, $\mathbb{H}$ denotes the hyperbolic plane over $\mathbb{Q}_2$.

Proof. See [[11], Theorem 3]. □

We assume that *Every $\mathbb{Z}$-lattice is primitive positive definite* throughout this paper. Because why if a $\mathbb{Z}$-lattice is indefinite, representation theory turns out to be easy one in most cases if we use spinor genus theory. For some $\mathbb{Z}$-lattice L the scale $\mathfrak{s}(L)$ is equal to $\mathbb{Z}$, then L is said to be primitive. The following theorem is very useful in the sequel.

Theorem 2.2.7. *For two $\mathbb{Z}$-lattices l and L, if $l_p {\to} L_p$ for all prime p (including ∞), then $l {\to} gen(L)$.*

Proof. See [[10], 102:5]. □

2.3 Representation theory of ternary form

In this section, we introduce general tools for the representation of arithmetic progression by a ternary quadratic form. We apply the method developed in [4] and [6].

Let d and a be a positive and non negative integer, respectively. We define

$$S_{d,a} = \{dn + a \mid n \in \mathbb{N} \cup \{0\}\}.$$

For ternary $\mathbb{Z}$-lattices M, N on the quadratic space V, we define

$$R(N, d, a) = \{x \in N/dN \mid Q(x) \equiv a \pmod{d}\}$$

and

$$R(M, N, d) = \{\sigma \in O(V) \mid \sigma(dN) \subset M\}.$$

A coset (or, a vector in the coset) $v \in R(N, d, a)$ is said to be *good* with respect to M, N, d and a if for any $x \in N$ satisfying $x \equiv v \pmod{d}$, there is a $\sigma \in R(M, N, d)$ such that $\sigma(x) \in M$. If every vector contained in $R(N, d, a)$ is good, we write

$$N \prec_{d,a} M.$$

The set of all good vectors in $R(N, d, a)$ is denoted by $R_M(N, d, a)$.
If $N \prec_{d,a} M$, then by Lemma 2.2 of [4], we have

$$S_{d,a} \cap Q(N) \subset Q(M).$$

Note that the converse is not true in general.

The following theorem is a little modified version of Corollary 2.2 of [6]:

Theorem 2.3.1. *Under the same notations given above, assume that there is a partition* $R(N, d, a) - R_M(N, d, a) = (P_1 \cup \cdots \cup P_k) \cup (\widetilde{P}_1 \cup \cdots \cup \widetilde{P}_{k'})$ *satisfying the following properties: for each* P_i, *there is a* $\tau_i \in O(V)$ *such that*

(i) τ_i *has an infinite order;*

(ii) $\tau_i(dN) \subset N$;

(iii) for any $v \in P_i$, $\tau_i(v) \in N$ *and* $\tau_i(v) \pmod{d} \in P_i \cup R_M(N, d, a)$,

and for each $\widetilde{P}_j$, *there is a* $\widetilde{\tau}_j \in O(V)$ *such that*

(iv) $\widetilde{\tau}_j(dN) \subset N$;

(v) for any $v \in \widetilde{P}_j$, $\widetilde{\tau}_j(v) \in N$ *and* $\widetilde{\tau}_j(v) \pmod{d} \in P_1 \cup \cdots \cup P_k \cup R_M(N, d, a)$.

Then we have

$$(S_{d,a} \cap Q(N)) - \bigcup_{i=1}^{k} Q(z_i)\mathbb{Z}^2 \subset Q(M),$$

where the vector z_i is a primitive vector in N which is an eigenvector of τ_i.

Proof. From the conditions (iv) and (v), we may assume that

$$R(N,d,a) - R_M(N,d,a) = P_1 \cup \cdots \cup P_k.$$

Hence the theorem follows directly from Corollary 2.2 in [6]. □

Lemma 2.3.2. *Let $w(L) = \sum_{[L'] \in gen(L)} \frac{1}{o(L')}$, where $[L']$ is the class of L'. Then*

$$\frac{1}{w(L)} \sum_{[L'] \in gen(L)} \frac{r(m,L')}{o(L')} = c^* \prod_p \alpha_p(m, L'_p),$$

where c^* can easily be computable and α_p is the local density depending only on the local structure of L over $\mathbb{Z}_p$.

Proof. See Theorem 6.8.1 in [13]. □

Lemma 2.3.3. *Let L be a ternary $\mathbb{Z}$-lattice with $h(L) > 1$. Assume that the genus of L contains only one spinor genus. If an integer a is represented by all isometric classes in the genus of L except at most only one isometric class, then for any positive integer $s > 1$ with $(s, dL) = 1$, as^2 is represented by all lattices in the genus of L.*

Proof. Without loss of generality, we may assume that there is a $\mathbb{Z}$-lattice, say M, that does not represent a. For any prime p not dividing dL, the graph $Z(p, L)$ defined in [12] is connected. Hence there is a $\mathbb{Z}$-lattice M' such that M' is adjacent to M and a is represented by M'. Since pM' is represented by M, ap^2 is represented by M. This completes the proof. □

Every computation such as $R(N,d,a)$ and $R_M(N,d,a)$ for some M, N, d and a was done by the computer program MAPLE.

Chapter 3

Universal sums of polygonal numbers

3.1 Sums of polygonal numbers and representation of ternary forms.

Let a, b, c be positive integers and $i, j, k \in \mathbb{N}$ such that $1 \leq i \leq j \leq k$ and at least one of j and k is greater than 2. A ternary sum

$$\Phi_{i,j,k}^{a,b,c}(x, y, z) = ap_{i+2}(x) + bp_{j+2}(y) + cp_{k+2}(z)$$

of generalized polygonal numbers is universal over $\mathbb{Z}$ if $\Phi_{i,j,k}^{a,b,c}(x, y, z) = n$ has an integer solution for any non negative integer n. Since the set of generalized hexagonal numbers is equal to the set of triangular numbers, we always assume that all of i, j and k is not 4. One may easily show that $\Phi_{i,j,k}^{a,b,c}$ is universal over $\mathbb{Z}$ if and only if the equation

$$\begin{aligned} &jka(2ix-(i-2))^2+ikb(2jy-(j-2))^2+ijc(2kz-(k-2))^2 \\ &\quad = 8ijkn + jka(i-2)^2 + ikb(j-2)^2 + ijc(k-2)^2 =: \Psi_{i,j,k}^{a,b,c}(n) \end{aligned} \tag{3.1.1}$$

has a solution over $\mathbb{Z}$ for every nonnegative integer n. Note that this is equivalent to the following diophantine equation

$$jkaX^2 + ikbY^2 + ijcZ^2 = \Psi^{a,b,c}_{i,j,k}(n) \tag{3.1.2}$$

has an integer solution X, Y and Z such that

$$X \equiv -i + 2 \pmod{2i},\ Y \equiv -j + 2 \pmod{2j} \text{ and } Z \equiv -k + 2 \pmod{2k}. \tag{3.1.3}$$

Therefore the problem is reduced to the representation of integers by ternary quadratic forms whose solution satisfies some congruence conditions. In many cases, the problem can be reduced to the problem of representations of integers by subforms of given forms without congruence condition on the solution. The following lemma reveals this correspondence more concretely in our case.

Lemma 3.1.1. *Assume that i, j, k and a, b, c given above satisfy the followings:*

(i) $\{i, j, k\} \subset \mathcal{S}_P = \{1, 2, p^n, 2p^n : p \text{ is an odd prime and } n \geq 1\}$*;*

(ii) all of (i, jka), (j, ikb) and (k, ijc) are one or two;

(iii) either i, j, k are all even or some of them are odd and

$$\Psi^{a,b,c}_{i,j,k}(n) \not\equiv 0 \pmod{2^{3+ord_2(ijk)}}.$$

Then there is a quadratic form $Q(x, y, z)$ such that for any integer n, $Q(x, y, z) = \Psi^{a,b,c}_{i,j,k}(n)$ has an integer solution if and only if Equation (3.1.2) satisfying (3.1.3) has an integer solution.

Proof. Note that for any $s \in \mathcal{S}_P$ and any integer a with $(a, s) = 1$, the congruence equation $x^2 \equiv a^2 \pmod{s}$ has at most two incongruent integer solutions modulo s.

First, assume that all of i, j and k are even. From the assumption, we have $i \equiv j \equiv k \equiv 2 \pmod 4$. Define

$$Q(x, y, z) = 16(jkaX^2 + ikbY^2 + ijcZ^2).$$

Assume that $(u, v, w) \in \mathbb{Z}^3$ is an integer solution of $Q(x, y, z) = \Psi^{a,b,c}_{i,j,k}(n)$. Then

$$16jkau^2 \equiv jka(i-2)^2 \pmod{32i}.$$

From this and the conditions (i) and (ii) follows $4u \equiv \pm(i-2) \pmod{2i}$. Similarly, we have $4v \equiv \pm(j-2) \pmod{2j}$ and $4k \equiv \pm(k-2) \pmod{2k}$. Therefore by choosing the signs suitably, we may find an integer solution of Equation (3.1.2) satisfying (3.1.3). For the converse, note that any integers X, Y, Z satisfying the condition (3.1.3) are divisible by 4.

Assume that exactly one of i, j, k, say i, is odd. Define

$$Q(x, y, z) = jkaX^2 + 16(ikbY^2 + ijcZ^2).$$

Assume that $(u, v, w) \in \mathbb{Z}^3$ is an integer solution of $Q(x, y, z) = \Psi^{a,b,c}_{i,j,k}(n)$. Since

$$16ikbv^2 \equiv ikb(j-2)^2 \pmod{j} \quad \text{and} \quad 16ijcw^2 \equiv ijc(k-2)^2 \pmod{k},$$

and $(j, ikb) = (k, ijc) = 2$ from (ii), we have

$$16v^2 \equiv (j-2)^2 \left(\text{mod } \frac{j}{2}\right) \quad \text{and} \quad 16w^2 \equiv (k-2)^2 \left(\text{mod } \frac{k}{2}\right),$$

which implies that $4v \equiv \pm(j-2) \pmod{2j}$ and $4k \equiv \pm(k-2) \pmod{2k}$. Now since $jkau^2 \equiv jka(i-2)^2 \pmod{32i}$ and $a \not\equiv 0 \pmod{8}$ by the condition (iii), we have $u^2 \equiv (i-2)^2 \pmod{2i}$. Hence $u \equiv \pm(i-2) \pmod{2i}$. Therefore by choosing the signs suitably, we may find an integer solution of (3.1.2) satisfying (3.1.3).

Assume that exactly two of i, j, k, say j, k, are odd. Define

$$Q(x, y, z) = 16jkaX^2 + ikbY^2 + ijc(Y - 4Z)^2.$$

Assume that $(u, v, w) \in \mathbb{Z}^3$ is an integer solution of $Q(x, y, z) = \Psi^{a,b,c}_{i,j,k}(n)$.

Then, similarly to the above, we may easily show that

$$4u \equiv \pm(i-2) \left(\text{mod } \frac{i}{2}\right), \ v \equiv \pm(j-2) \pmod{j}, \ v-4w \equiv \pm(k-2) \pmod{k}.$$

Since $i \equiv 2 \pmod 4$, we have $4u \equiv \pm(i-2) \pmod{2i}$. Furthermore, since

$$ikbv^2 + ijc(v-4w)^2 \equiv ikb + ijc \not\equiv 0 \ (\text{mod } 16),$$

v is odd. Hence

$$v \equiv \pm(j-2) \ (\text{mod } 2j), \ v-4w \equiv \pm(k-2) \ (\text{mod } 2k).$$

Therefore by choosing the signs suitably, we may find an integer solution of (3.1.2) satisfying (3.1.3). Conversely, if Equation (3.1.2) satisfying (3.1.3) has an integer solution X, Y, Z, then X is divisible by 4 and both Y and Z re odd. Hence $Y \equiv Z \pmod 4$ or $Y \equiv -Z \pmod 4$. This implies that $Q(x,y,z) = \Psi_{i,j,k}^{a,b,c}(n)$ has an integer solution.

Finally, if all of i, j, k are odd, then we define

$$Q(x,y,z) = jkaX^2 + ikb(X-4Y)^2 + ijc(X-4Z)^2.$$

The proof of the remaining part is quite similar to the above. □

The following corollaries are classifications of representations of quadratic forms with some congruence conditions corresponding to representations of a subform.

Corollary 3.1.2. *Let n be any positive integer. In the following, the left hand side has an integer solution if and only if the right hand side has an integer* *solution:*

$$\begin{aligned}
P_3(x) + 2P_4(y) + 4P_5(z) = n &\iff 3X^2 + 4Y^2 + 48Z^2 = 24n + 7, \\
P_3(x) + 4P_4(y) + 2P_5(z) = n &\iff 2X^2 + 3Y^2 + 96Z^2 = 24n + 5, \\
P_3(x) + P_4(y) + 2P_7(z) = n &\iff 2X^2 + 5Y^2 + 40Z^2 = 40n + 23, \\
P_3(x) + 2P_3(y) + 2P_8(z) = n &\iff 3X^2 + 6Y^2 + 16Z^2 = 24n + 25. \\
P_3(x) + 2P_3(y) + P_{12}(z) = n &\iff 5X^2 + 8Y^2 + 10Z^2 = 40n + 47.
\end{aligned}$$

Proof. Since the proofs are quite similar to each other, we only consider the first and the last case.

By Equation (3.1.2), $P_3(x) + 2P_4(y) + 4P_5(z) = n$ has an integer solution if and only if

$$f(X, Y, Z) = 3X^2 + 4Y^2 + 48Z^2 = 24n + 7$$

has a solution such that

$$X \equiv 1 \pmod 2 \quad \text{and} \quad Y \equiv 1, 5 \pmod 6.$$

Let (X_0, Y_0, Z_0) be an integer solution. Then clearly, $X_0 \equiv Y_0 \equiv 1 \pmod 2$, for

$$3X_0^2 + 4Y_0^2 \equiv 7 \pmod 8 \ \text{ and } \ X_0^2, Y_0^2 \equiv 0, 1, 4 \pmod 8.$$

Since $3X_0^2 + 4Y_0^2 \equiv 1 \pmod 6$ and $X_0^2, Y_0^2 \equiv 1, 3, 4 \pmod 6$, Y_0 is relatively prime to 6. Hence, $P_3(x) + 2P_4(y) + 4P_5(z) = n$ has an integer solution if and only if the integer $24n + 7$ is represented by the ternary quadratic form $3X^2 + 4Y^2 + 48Z^2$.

Now consider the diophantine equation $P_3(x) + 2P_3(y) + P_{12}(z) = n$. By Equation (3.1.2), it can be reduced to the representation of the quadratic form

$$5X^2 + 8Y^2 + 10Z^2 = 40n + 47,$$

where $X \equiv Z \equiv 1 \pmod 2$ and $Y \equiv 2$ or $3 \pmod 5$. If it has an integer solution (X, Y, Z), then X and Z should be odd, for

$$5X^2 + 2Z^2 \equiv 7 \pmod 8 \quad \text{and} \quad X^2, Z^2 \equiv 0, 1, 4 \pmod 8.$$

Moreover, since $3Y^2 \equiv 2 \pmod 5$, we have $Y \equiv 2$ or $3 \pmod 5$. Therefore, $P_3(x) + 2P_3(y) + P_{12}(z) = n$ has an integer solution if and only if the integer $40n + 47$ is represented by the ternary quadratic form $5x^2 + 8y^2 + 10z^2$. This completes the proof. $\square$

Corollary 3.1.3. *Let n be any positive integer. In the following, the left hand side has an integer solution if and only if the right hand side has an integer solution:*

$$\begin{aligned}
P_3(x) + P_5(y) + P_7(z) = n &\iff 3(Z - 2X)^2 + 5Y^2 + 15Z^2 = 120n + 47,\\
P_3(x) + P_7(y) + P_8(z) = n &\iff 3(Y - 2X)^2 + 15Y^2 + 40Z^2 = 120n + 202.\\
P_3(x) + 2P_3(y) + P_9(z) = n &\iff X^2 + 7(Z - 2Y)^2 + 14Z^2 = 56n + 46,\\
P_3(x) + P_3(y) + P_{12}(z) = n &\iff 5(Y - 2X)^2 + 5Y^2 + 8Z^2 = 40n + 42.
\end{aligned}$$

Proof. Since the proofs are quite similar to each other, we only consider the first case. By Equation (3.1.2), if $P_3(x) + P_5(y) + P_7(z) = n$ has an integer solution if and only if the ternary quadratic form

$$3X^2 + 5Y^2 + 15Z^2 = 120n + 47$$

has an integer solution X, Y and Z such that

$$X \equiv 3 \text{ or } 7 \pmod{10}, \quad Y \equiv 1 \text{ or } 5 \pmod 6 \quad \text{and} \quad Z \equiv 1 \pmod 2.$$

If the quadratic form

$$3X^2 + 5Y^2 + 15Z^2 = 120n + 47$$

has an integer solution, then one may easily check that

$$(X, Y, Z) \equiv (0, 0, \pm 1), (2, 2, \pm 1), (\pm 1, 2, 0), (\pm 1, 0, 2), (\pm 1, \pm 1, \pm 1) \pmod 4.$$

Hence if $X \equiv Z \pmod 2$, then X, Y and Z are all odd. Therefore we consider the following diophantine equation

$$3(Z-2X)^2 + 5Y^2 + 15Z^2 = 120n + 47.$$

If it has an integer solution, then one may easily show that $P_3(x) + P_5(y) + P_7(z) = n$ has an integer solution. $\square$

Corollary 3.1.4. *Let n be any positive integer, In the followings, the left hand side has an integer solution if and only if the right hand side has an integer solution:*

$$\begin{aligned}
P_3(x) + 9P_3(y) + P_5(z) = n &\iff (Y-4X)^2 + 3Y^2 + 27Z^2 = 24n + 31, \\
P_3(x) + 2P_5(y) + P_7(z) = n &\iff 3(Z-4X)^2 + 10Y^2 + 15Z^2 = 120n + 52, \\
2P_3(x) + P_5(y) + P_9(z) = n &\iff 3(Y-4X)^2 + 7Y^2 + 42Z^2 = 168n + 124.
\end{aligned}$$

Proof. By Equation (3.1.2), $P_3(x)+9P_3(y)+P_5(z) = n$ has an integer solution if and only if

$$f(X,Y,Z) = X^2 + 3Y^2 + 27Z^2 = 24n + 31$$

has a solution such that

$$Y \equiv Z \equiv 1 \pmod 2 \quad \text{and} \quad X \equiv 1, 5 \pmod 6.$$

Consider the following ternary form

$$g(X,Y,Z) = (Y-4X)^2 + 3Y^2 + 27Z^2,$$

which is a subform of the above ternary form. Suppose that $g(X,Y,Z) = 24n+31$ has an integer solution. Let (X_0, Y_0, Z_0) be an integer solution. Then clearly, $X_0 \equiv Y_0 \equiv Z_0 \equiv 1 \pmod 2$ and $Y_0 - 4X_0 \not\equiv 0 \pmod 3$. Therefore

$$Y = Y_0,\ Z = Z_0,\ X = Y_0 - 4X_0$$

is an integer solution of $f(X,Y,Z) = 24n+31$ satisfying the above congruence conditions. The proof of the remaining cases are quite similar to the above. □

Remark 3.1.5. Among 33 candidates given in the introduction, those who do not satisfy the conditions in Lemma 3.1.1 are the cases

$$4,\ 7,\ 9,\ 14,\ 16,\ 24,\ \text{and } 27 \sim 32.$$

For 4-th, 9-th and 14-th cases among them, we prove their universalities by using some other methods. For details, see Section 4.

Chapter 4

Ternary universal sums of polygonal numbers

In this section, we consider the conjecture given by Sun in [8] on the ternary universal sums of generalized polygonal numbers. Among remaining 33 candidates of ternary universal sums of generalized polygonal numbers, we prove the universality of 15 candidates. Main ingredient is to reduce the problem to the representation of ternary quadratic forms without any congruence condition, which is introduced in Chapter 3.

Theorem 4.0.6. *The following* 15 *ternary sums of generalized polygonal numbers are all universal:*

$$\begin{array}{lll}
P_3(x)+2P_4(y)+4P_5(z), & P_3(x)+4P_4(y)+2P_5(z), & P_3(x)+P_4(y)+2P_7(z), \\
P_3(x)+2P_3(y)+2P_8(z), & P_3(x)+2P_3(y)+P_{12}(z), & P_3(x)+P_5(y)+P_7(z), \\
P_3(x)+P_7(y)+P_8(z), & P_3(x)+2P_3(y)+P_9(z), & P_3(x)+P_3(y)+P_{12}(z), \\
P_3(x)+9P_3(y)+P_5(z), & P_3(x)+2P_5(y)+P_7(z), & 2P_3(x)+P_5(y)+P_9(z), \\
P_3(x)+2P_3(y)+P_7(z), & P_3(x)+P_7(y)+2P_7(z), & P_3(x)+P_4(y)+P_9(z).
\end{array}$$

Proof. In each case, we will use Theorem 2.3.1 to prove the universality. In each case, we provide the data

$$(P_1,\tau_1),\ (P_2,\tau_2),\ldots,(P_k,\tau_k)$$

for some suitable $\mathbb{Z}$-lattices M and N.

(1) $P_3(x) + 2P_4(y) + 4P_5(z)$

By Corollary 3.1.2, it suffices to show that the diophantine equation

$$3X^2 + 4Y^2 + 48Z^2 = 24n + 7$$

has an integer solution X, Y and Z for any nonnegative integer n. Let

$$M = \langle 3, 4, 48\rangle, \quad N = \langle 3, 4, 12\rangle.$$

Note that $S_{24,7} \subset Q(\mathrm{gen}(N))$ and $h(N) = 1$, we have $S_{24,7} \subset Q(N)$. One may easily check that

$$R(N, 4, 3) - R_M(N, 4, 3) = \pm\{(1, 0, 1), (1, 2, 3), (1, 0, 3), (1, 2, 1) \ (\mathrm{mod} 4)\}.$$

Let $P_1 = \pm\{(1, 0, 1), (1, 2, 3) \ (\mathrm{mod} 4)\}$, $\widetilde{P}_1 = \pm\{(1, 0, 3), (1, 2, 1) \ (\mathrm{mod} 4)\}$ and

$$\tau_1 = \frac{1}{4}\begin{pmatrix} 0 & -4 & -4 \\ 3 & 1 & -3 \\ -1 & 1 & -3 \end{pmatrix}, \quad \widetilde{\tau}_1 = \frac{1}{4}\begin{pmatrix} 0 & 4 & 4 \\ 3 & -1 & 3 \\ -1 & -1 & 3 \end{pmatrix}.$$

Then one may easily show that (P_1, τ_1) and $(\widetilde{P}_1, \widetilde{\tau}_1)$ satisfies all conditions of Theorem 2.3.1. The eigenvector of τ_1 is $z_1 = (1, 0, 1)$ and $Q(z_1) = 15$ is not of the form $24n + 7$. Hence we have $S_{24,7} \subset Q(M)$. This implies that the equation $3X^2 + 4Y^2 + 48Z^2 = 24n + 7$ has an integer solution X, Y, Z for every non negative integer n. We know that $P_3(x) + 2P_4(y) + 4P_5(z)$ is universal over $\mathbb{Z}$.

(2) $P_3(x) + 4P_4(y) + 2P_5(z)$

By Corollary 3.1.2, it suffices to show that the diophantine equation

$$2X^2 + 3Y^2 + 96Z^2 = 24n + 5$$

has an integer solution X, Y and Z for any nonnegative integer n.

Let

$$M = \langle 2, 3, 96 \rangle, \quad N = \begin{pmatrix} 5 & 1 & 1 \\ 1 & 11 & -1 \\ 1 & -1 & 11 \end{pmatrix} \quad \text{and} \quad N' = \langle 2 \rangle \perp \begin{pmatrix} 12 & 6 \\ 6 & 27 \end{pmatrix}.$$

Note that $\text{gen}(M) = \{[M], [N], [N']\}$. $S_{24,5}$ is an eligible integer of gen(M) for any nonnegative integer n and represented by M, N and N'. By direct computation, $N' \prec_{4,1} M$. Hence we know that

$$S_{4,1} \cap Q(N') \subset Q(M).$$

For N, we have

$$R(N, 12, 5) - R_M(N, 12, 5) = \pm\big\{(3\ 1\ 7), (3\ 2\ 8), (3\ 4\ 10), (3\ 5\ 11), \\ (3\ 7\ 1), (3\ 8\ 2), (3\ 10\ 4), (3\ 11\ 5)\ (\text{mod}12)\big\}.$$

Let

$$P_1 = R(N, 12, 5) - R_M(N, 12, 5) \quad \text{and} \quad \tau_1 = \begin{pmatrix} 8 & 10 & -10 \\ -6 & -3 & -9 \\ 2 & -11 & -1 \end{pmatrix}.$$

Then one may easily show that (P_1, τ_1) satisfy all conditions of Theorem 2.3.1. Since the eigenvector of τ_1 is $z_1 = (0, 1, 1)$ and $Q(z_1) = 20$, $24n+5$ is represented by M. The quadratic form $2X^2 + 3Y^2 + 96Z^2 = 24n + 5$. has an integer solution X, Y and Z. Therefore, the sum $P_3(x)+4P_4(y)+2P_5(z)$ is universal over $\mathbb{Z}$.

(3) $P_3(x) + P_4(y) + 2P_7(z)$

By Corollary 3.1.2, it suffices to show that the diophantine equation

$$2X^2 + 5Y^2 + 40Z^2 = 40n + 23.$$

has an integer solution X, Y and Z for any nonnegative integer n. Let

$$M = \langle 2, 5, 40 \rangle \quad \text{and} \quad N = \langle 5, 8, 10 \rangle.$$

Note that

$$\text{gen}(M) = \{[M], [N]\}.$$

One may easily compute the set $R(N, 8, 7) - R_M(N, 8, 7)$ which consists of following vectors:

$$\begin{array}{llllll} \pm(1,0,1), & \pm(1,0,7), & \pm(1,2,3), & \pm(1,2,5), & \pm(1,4,1), & \pm(1,4,7), \\ \pm(1,6,3), & \pm(1,6,5), & \pm(3,0,3), & \pm(3,0,5), & \pm(3,2,1), & \pm(3,2,7), \\ \pm(3,4,3), & \pm(3,4,5), & \pm(3,6,1), & \pm(3,6,7). & & \end{array}$$

Let

$$R(N, 8, 7) - R_M(N, 8, 7) = P_1 \cup P_2,$$

where

$$P_1 = \{(x, y, z) \mid x \equiv z \pmod 4\} \quad \text{and}$$

$$P_2 = \{(x, y, z) \mid x + z \equiv 0 \pmod 4\}$$

with $x \equiv z \equiv 1 \pmod 2$ and $y \equiv 0 \pmod 2$.

If we let

$$\tau_1 = \frac{1}{8} \begin{pmatrix} 4 & 8 & 4 \\ -5 & 2 & 5 \\ 2 & -4 & 6 \end{pmatrix} \quad \text{and} \quad \tau_2 = \frac{1}{8} \begin{pmatrix} 4 & 8 & -4 \\ -5 & 2 & -5 \\ -2 & 4 & 6 \end{pmatrix}.$$

then (P_1, τ_1) and (P_2, τ_2) satisfies all conditions in the Theorem 2.3.1. The eigenvectors of τ_1 and τ_2 are $z_1 = (1, 0, 1)$ and $z_1 = (1, 0, -1)$, respectively. However, $Q(z_1) = Q(z_2) = 20$ is not the form of $40n + 23$. Therefore $(S_{8,7} \cap Q(N)) - 15\mathbb{Z}^2 \subset Q(M)$. Since $S_{40,23} \subset Q(\text{gen}(M))$, we have $S_{40,23} \subset Q(M)$. Hence, the quadratic form $2x^2 + 5y^2 + 40z^2 = 40n + 23$. has an integer solution X, Y and Z. The sum $P_3(x) + P_4(y) + 2P_7(z)$ is universal over $\mathbb{Z}$.

(4) $P_3(x) + 2P_3(y) + 2P_8(z)$

By Corollary 3.1.2, it suffices to show that the diophantine equation

$$3X^2 + 6Y^2 + 16Z^2 = 24n + 25$$

has an integer solution X, Y and Z for any nonnegative integer n. Let

$$M = \langle 3, 6, 16 \rangle \quad \text{and} \quad N = \langle 1 \rangle \perp \begin{pmatrix} 18 & 6 \\ 6 & 18 \end{pmatrix}.$$

Note that

$$\text{gen}(M) = \{[M], [N]\}$$

and $24n + 1$ is an eligible integer for $\text{gen}(M)$. We can easily calculate that

$$R(N, 3, 1) - R_M(N, 3, 1) = \pm\{(1, 0, 0) \pmod 3\}.$$

Let $P_1 = \pm\{(1, 0, 0) \pmod 3\}$ and

$$\tau_1 = \frac{1}{3}\begin{pmatrix} 3 & 0 & 0 \\ 0 & -2 & -3 \\ 0 & 3 & 0 \end{pmatrix}.$$

Then one may easily show that this information satisfies all conditions in the Theorem 2.3.1. The eigenspace of τ_1 is $z_1 = (t, 0, 0)$ and $Q(z_1) = t^2$, that is, every positive integer of the form $24n + 21$ is represented by M except t^2. Since N is contained in the spinor genus of M and 1 is represented by N, every square of an integer that has a prime divisor bigger than 3 is represented by both M and N by Lemma 2.3.3. Therefore every integer of the form $24n + 1$ for some positive integer n is represented by M. Therefore the sum $P_3(x) + 2P_3(y) + 2P_8(z)$ is universal over $\mathbb{Z}$.

(5) $P_3(x) + 2P_3(y) + P_{12}(z)$

By Corollary 3.1.2, it suffices to show that

$$5X^2 + 8Y^2 + 10Z^2 = 40n + 47$$

has an integer solution X, Y and Z for any nonnegative integer n. Let $M = \langle 5, 8, 10\rangle$. Note that

$$\text{gen}(M) = \{[M], [N = \langle 2, 5, 40\rangle]\}$$

and $40n + 7$ is an eligible integer for gen(M). One may easily compute that

$$\begin{aligned} R(N, 8, 7) - R_M(N, 8, 7) = \pm\{&(1, 1, 0), (1, 1, 4), (1, 5, 2), (1, 5, 6), (3, 3, 0),\\ &(3, 3, 4), (3, 7, 2), (3, 7, 6), (1, 3, 2), (1, 3, 6),\\ &(1, 7, 0), (1, 7, 4), (3, 1, 2), (3, 1, 6), (3, 5, 0),\\ &(3, 5, 4) \pmod 8\}. \end{aligned}$$

Let

$$P_1 = \{(x, y, z) \mid x - y + 2z \equiv 0 \ (\text{mod } 8)\}$$

and

$$P_2 = \{(x, y, z) \mid x + y + 2z \equiv 0 \ (\text{mod } 8)\}.$$

Note that

$$R(N, 8, 7) - R_M(N, 8, 7) = P_1 \cup P_2.$$

If we define

$$\tau_1 = \begin{pmatrix} 2 & -10 & 20 \\ -4 & -4 & -8 \\ -1 & 1 & 6 \end{pmatrix} \quad \text{and} \quad \tau_2 = \begin{pmatrix} 2 & 10 & 20 \\ 4 & -4 & 8 \\ -1 & -1 & 6 \end{pmatrix},$$

then one may easily show that (P_1, τ_1) and (P_2, τ_2) satisfies all conditions in Theorem 2.3.1. The eigenspaces of both τ_1 and τ_2 are $(t, t, 0)$ and $f_N(t, t, 0) = 7t^2$. However there is an integer in the arithmetic progression $40n+7$ which is of the form $7t^2$. Note that if a prime p divide t, then $p \geq 7$. By Minkowski-Siegel formula, we know that if p is a prime greater than 7,

$$r(7p^2, \langle 5,8,10\rangle) + r(7p^2, \langle 2,5,40\rangle) = 4p + 4 - 4 \cdot \left(\frac{-7}{p}\right).$$

Furthermore, if $p = 7$ then we know that

$$r(7^3, \langle 5,8,10\rangle) = 24.$$

Then there is a representation which is not contained in the eigenspace for both cases. Therefore, $P_3(x) + 2P_3(y) + P_{12}(z)$ is universal over $\mathbb{Z}$.

(6) $P_3(x) + P_5(y) + P_7(z)$

By Corollary 3.1.3, it suffices to show that the diophantine equation

$$\begin{aligned} 3(X-2Z)^2 + 5Y^2 + 15Z^2 &= 12X^2 - 12ZX + 18Z^2 + 5Y^2 \\ &= 120n + 47 \end{aligned}$$

has an integer solution for any nonnegative integer n. Let

$$M = \langle 5\rangle \perp \begin{pmatrix} 12 & 6 \\ 6 & 18 \end{pmatrix}.$$

Note that $\text{gen}(M) = \{[M], [\langle 2,15,30\rangle]\}$ and every integer of the form $120n + 47$ is an eligible integer for $\text{gen}(M)$ for any nonnegative integer n.

Let N be a $\mathbb{Z}$-lattice such that

$$f_N(x,y,z) = 2x^2 + 15y^2 + 30z^2.$$

One may compute that

$$P_1 = R(N,4,3) - R_M(N,4,3) = \{(0,\pm 1,0), (2,\pm 1,2)\}.$$

If we take

$$\tau_1 = \frac{1}{4}\begin{pmatrix} 1 & 0 & 15 \\ 0 & -4 & 0 \\ -1 & 0 & 1 \end{pmatrix},$$

then one may easily check that P_1 and τ_1 satisfy all conditions of Theorem 2.3.1. Finally, since the eigenspace of τ_1 is $z_1 = (0, t, 0)$, and $f_N(z_1) = 15t^2$ is not of the form $120n + 47$, every integer of the form $4n + 3$ that is represented by N is also represented by M. Therefore the eligible number of the form $120n + 47$ for $\text{gen}(M)$ is represented by M, which implies that $P_3(x) + P_5(y) + P_7(z)$ is universal over $\mathbb{Z}$.

(7) $P_3(x) + P_7(y) + P_8(z)$

By Corollary 3.1.3, it suffices to show that the diophantine equation

$$f(X, Y, Z) = 3(Y - 2X)^2 + 15Y^2 + 40Z^2 = 120n + 202$$

has an integer solution for any nonnegative integer n.

Let

$$M = \begin{pmatrix} 6 & 3 \\ 3 & 9 \end{pmatrix} \perp \langle 20 \rangle, \quad N = \langle 5 \rangle \perp \begin{pmatrix} 9 & 3 \\ 3 & 21 \end{pmatrix}.$$

Note that $\text{gen}(M) = \{[M], [N]\}$.

Then we have

$$P_1 = R(N, 6, 5) - R_M(N, 6, 5) = \pm \{(1, 0, 0), (1, 3, 3) \pmod 6\}.$$

If let

$$\tau_1 = \frac{1}{6}\begin{pmatrix} 6 & 0 & 0 \\ 0 & -5 & -7 \\ 0 & 3 & -3 \end{pmatrix},$$

Then one may easily show that (P_1, τ_1) satisfies all conditions of Theorem 2.3.1. Note that the eigenspace of τ_1 is $z_1 = (t, 0, 0)$ and $Q(z_1) =$

$5t^2$. It is not the form of $60n+41$. The quadratic form $6X^2-6XY+9Y^2+20Z^2=60n+41$ has an integer solution X, Y and Z. Since f is isometric to $2M$.

$$f(X,Y,Z)=3(Y-2X)^2+15Y^2+40Z^2=120n+202$$

has an integer solution for any nonnegative integer n. We also directly compute that $N \prec_{30,11} M$. Since $S_{60,41} \subset Q(\text{gen}(M))$, we have $S_{60,41} \subset Q(M)$. Therefore, the sum $P_3(x)+P_7(y)+P_8(z)$ is universal over $\mathbb{Z}$.

(8) $P_3(x)+2P_3(y)+P_9(z)$

By Corollary 3.1.3, it suffices to show that

$$X^2+7(Z-2Y)^2+14Z^2=56n+46$$

has an integer solution X, Y and Z for any nonnegative integer n. To show this, we first use the fact that $\text{gen}(\langle 1,7,14\rangle)=\{[\langle 1,7,14\rangle],[\langle 2,7,7\rangle]\}$ and $\langle 2,7,7\rangle \prec_{8,6} \langle 1,7,14\rangle$. Hence one easily check that every positive integer of the form $56n+46$ is represented by $\langle 1,7,14\rangle$. Let

$$M=\langle 1\rangle \perp \begin{pmatrix} 21 & 7 \\ 7 & 21 \end{pmatrix} \quad \text{and} \quad N=\langle 1,7,14\rangle.$$

One may easily show that

$$\begin{aligned} P_1 &= R(N,8,6)-R_M(N,8,6) \\ &= \{(0,0,\pm 1),(0,0,\pm 3),(4,4,\pm 1),(4,4,\pm 3) \pmod 8\} \end{aligned}$$

One may easily show that this set and

$$\tau_1=\frac{1}{8}\begin{pmatrix} 1 & 21 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & -8 \end{pmatrix}$$

satisfy the condition in Theorem 2.3.1. Note that the eigenvector of τ_1 is $z_1 = (0, 0, 1)$ and $Q(z_1) = 14$. Since every integer of the form $56n + 46$ is not divisible by 14, it is represented by M. Therefore $P_3(x) + 2P_3(y) + P_9(z)$ is universal over $\mathbb{Z}$.

(9) $P_3(x) + P_3(y) + P_{12}(z)$

By Corollary 3.1.3, it suffices to show that

$$f(X, Y, Z) = 5(Y - 2X)^2 + 5Y^2 + 8Z^2 = 40n + 42$$

has an integer solution X, Y and Z for any nonnegative integer n. Let

$$M = \langle 5, 5, 4\rangle \text{ and } N = \langle 1, 5, 20\rangle.$$

Then we have

$$R(N, 20, 1) - R_M(N, 20, 1) = \{(1, t, 5s), (9, t, 5s), (11, t, 5s), (19, t, 5s)\}, \quad \text{for } t = 0, 10 \text{ and } s = 0, 1, 2, 3.$$

Let

$$P_1 = \{(1, t, 5s), (9, t, 5s), (11, t, 5s), (19, t, 5s)\}, \quad \text{for } t = 0, 10 \text{ and } s = 0, 1, 2, 3.$$

and

$$\tau_1 = \frac{1}{20}\begin{pmatrix} 20 & 0 & 0 \\ 0 & -12 & 32 \\ 0 & -8 & 12 \end{pmatrix}.$$

Then one may easily show that this information satisfies all conditions of Theorem 2.3.1. Note that the eigenspace of τ_1 is $z_1 = (t, 0, 0)$ and $Q(z_1) = t^2$.

Therefore if $20n + 1$ is not of the form t^2 for a positive integer t, then it is represented by M. Since N is contained in the spinor genus of M and 1 is represented by N, every square of an integer that has a prime divisor relatively prime to 10 is represented by M by Lemma 2.3.3.

Therefore every integer of the form $20n+1$ for a positive integer n is represented by M.

Since f is isometric to $2M$, f has an has an integer solution X, Y and Z for any nonnegative integer n. Hence $P_3(x)+P_3(y)+P_{12}(z)$ is universal over $\mathbb{Z}$.

(10) $P_3(x)+9P_3(y)+P_5(z)$

By Corollary 3.1.4, it suffices to show that

$$(Y-4X)^2+3Y^2+27Z^2=24n+7$$

has a solution over $\mathbb{Z}$ for any nonnegative integer n. Equivalently, the quadratic form

$$4X^2+8XY+16Y^2+27Z^2=24n+7$$

has an integer solution X, Y and Z for any nonnegative integer n. We can easily see that

$$\begin{pmatrix} 4 & 4 \\ 16 & 4 \end{pmatrix} \perp \langle 27 \rangle \quad \simeq \quad \langle 4, 12, 27 \rangle.$$

Let

$$M = \langle 4, 12, 27 \rangle \quad \text{and} \quad N = \langle 3, 4, 12 \rangle.$$

Note that $h(N)=1$ and every positive integer of the form $24n+7$ is eligible integer of N. Therefore N represents every positive integer of the form $24n+7$. One may easily show that $R(N,24,7)-R_M(N,24,7)$ contains 128 vectors, and for any vector (v_1,v_2,v_3) in the set, $v_1 \equiv v_2 \equiv 1 \pmod 2$ and $v_3 \equiv 0 \pmod 6$. Let

$$R(N,24,7)-R_M(N,24,7)=P_1 \cup \widetilde{P}_1,$$

where

$$\begin{aligned} P_1 &= \{(v_1, v_2, v_3) \mid v_1v_2 \equiv 1 \pmod 3 \quad \text{and} \quad v_1 + v_2 + v_3 \equiv 0 \pmod 4\}, \\ \widetilde{P}_1 &= \{(v_1, v_2, v_3) \mid v_1v_2 \equiv -1 \pmod 3 \quad \text{and} \quad v_1 \equiv v_2 + v_3 \pmod 4\} \end{aligned}$$

and

$$\tau_1 = \frac{1}{24}\begin{pmatrix} 0 & 24 & 24 \\ 18 & -6 & 18 \\ -6 & -6 & 18 \end{pmatrix}, \qquad \widetilde{\tau}_1 = \frac{1}{24}\begin{pmatrix} 0 & -24 & -24 \\ 18 & 6 & -18 \\ -6 & 6 & -18 \end{pmatrix}.$$

Then (P_1, τ_1) and $(\widetilde{P}_1, \widetilde{\tau}_1)$ satisfies all conditions of Theorem 2.3.1. The eigenvector of τ_1 is $z_1 = (1, -1, 0)$ and $\widetilde{\tau}_1(-1, -1, 0) = (1, -1, 0)$. Even if $24a + 7$ is one of the $7t^2$, $24a + 7$ is represented by M except only when $r(7t^2, N)$ is below 4. Assume that $24a+7 = 7t^2$ for a nonnegative integer t and there is an odd prime p which is 5 or more dividing t. Since the class number of N is one, we have

$$r(7p^2, N) = 4p + 4 - 4 \cdot \left(\frac{-7}{p}\right) > 4$$

by the Minkowski-Siegel formula. There is a representation which is not contained in the eigenspace. Hence, $40n + 47$ is represented by M if t has an odd prime divisor. Therefore, the sum $P_3(x) + 9P_3(y) + P_5(z)$ is universal over $\mathbb{Z}$.

(11) $P_3(x) + 2P_5(y) + P_7(z)$

By Corollary 3.1.4, it suffices to show that

$$f(X, Y, Z) = 3(Z - 4X)^2 + 10Y^2 + 15Z^2 = 120n + 52.$$

has an integer solution X, Y and Z. for any nonnegative integer n. Let

$$M = \langle 5 \rangle \perp \begin{pmatrix} 9 & 3 \\ 3 & 21 \end{pmatrix}, \quad N = \begin{pmatrix} 6 & 3 \\ 3 & 9 \end{pmatrix} \perp \langle 20 \rangle.$$

Note that $\text{gen}(M) = \{[M], [N]\}$. One may easily compute that $N \prec_{15,11} M$. Since $S_{60,26} \subset Q(\text{gen}(M))$, we have $S_{60,26} \subset Q(M)$. Since f is isometric to $2M$, f has an integer solution X, Y, Z for any non negative integer n. Therefore, the sum $p_3(x) + 2p_5(y) + p_7(z)$ is universal over $\mathbb{Z}$.

(12) $2P_3(x) + P_5(y) + P_9(z)$

By Corollary 3.1.4, it suffices to show that

$$f(X, Y, Z) = 3(y-4x)^2 + 7y^2 + 42z^2 = 48x^2 - 24xy + 10y^2 + 42z^2 = 168n + 124$$

has an integer solution X, Y and Z for any nonnegative integer n. Let

$$M = \begin{pmatrix} 5 & 1 \\ 1 & 17 \end{pmatrix} \perp \langle 21 \rangle, \quad N = \begin{pmatrix} 12 & 0 & 6 \\ 0 & 14 & 7 \\ 6 & 7 & 17 \end{pmatrix}, \quad N' = \begin{pmatrix} 5 & 2 \\ 2 & 5 \end{pmatrix} \perp \langle 84 \rangle.$$

Note that $\text{gen}(M) = \{[M], [N], [N']\}$. One may easily compute that $N \prec_{21,20} M$ and $N' \prec_{21,20} M$. Since $S_{84,62} \subset Q(\text{gen}(M))$, we have $S_{84,62} \subset Q(M)$. Since f is isometric to $2M$, f equation has an integer solution X, Y, Z for any non negative integer n. Therefore, the sum $2p_3(x) + p_5(y) + p_9(z)$ is universal over $\mathbb{Z}$.

(13) $P_3(x) + 2P_3(y) + P_7(z)$

By 3.1.2 and 3.1.3, the ternary sum $P_3(x) + 2P_3(y) + P_7(z)$ is univeral over $\mathbb{Z}$ if and only if the quadratic form

$$f(X, Y, Z) = X^2 + 5Y^2 + 10Z^2 = 40n + 24$$

has an integer solution X, Y and Z with $\gcd(XYZ, 2) = 1$. Let

$$M = \langle 1, 5, 10 \rangle.$$

Note that $h(M) = 1$ and every positive integer of the form $40n + 24$ is eligible integer of M. Therefore M represents every positive integer of the form $40n + 24$. Let (x, y, z) be an integer solution. One may easily check that all of x, y and z have a same parity. Suppose that x, y and z are all even. Then $x \equiv y \pmod 4$ and there is another representation

$$F\left(\frac{x+5y+10z}{4}, \frac{-x+3y-2z}{4}, \frac{-x-y+2z}{4}\right) = F(x, y, z) = 40n+32.$$

Let

$$\tau_1 = \frac{1}{4}\begin{pmatrix} 1 & 5 & 10 \\ -1 & 3 & -2 \\ -1 & -1 & 2 \end{pmatrix}.$$

We know that τ_1 has an infinite order and there are only finitely many representations. This means that if $x \equiv 0 \pmod 2$, then there exist odd solution x' after several computations unless . Hence the quadratic form $X^2 + 5Y^2 + 10Z^2 = 40n + 24$ has an integer solution X, Y and Z with $\gcd(XYZ, 2) = 1$. Therefore, $P_3(x) + 2P_3(y) + P_7(z)$ is universal over $\mathbb{Z}$.

(14) $P_3(x) + P_7(y) + 2P_7(z)$

By 3.1.2 and 3.1.3, the ternary sum $P_3(x) + P_7(y) + 2P_7(z)$ is universal over $\mathbb{Z}$ if and only if the quadratic form

$$f(X, Y, Z) = X^2 + 2Y^2 + 5Z^2 = 40n + 32$$

has an integer solution X, Y and Z such that $X \equiv 0 \pmod 5), Y \equiv 1, 4 \pmod 5$ or $X \equiv 2, 3 \pmod 5), Y \equiv 2, 3 \pmod 5$.

First we will show that $X^2+2Y^2+5Z^2 = 40n+32$ has an integer solution X, Y and Z with $X \equiv Y \pmod 5$ if and only if $(Y-5X)^2+2Y^2+5Z^2 =$

$40n + 32$ has an integer solution X, Y and Z. We can easily see that

$$\begin{pmatrix} 3 & 5 \\ 5 & 25 \end{pmatrix} \perp \langle 5 \rangle \simeq \begin{pmatrix} 3 & 1 \\ 1 & 17 \end{pmatrix} \perp \langle 5 \rangle.$$

Let

$$M = \begin{pmatrix} 3 & 1 \\ 1 & 17 \end{pmatrix} \perp \langle 5 \rangle.$$

Note that

$$gen(M) = \{[M], [N = \langle 2, 5, 25 \rangle]\}.$$

Then We can compute that

$$N \prec_{8,0} M.$$

Therefore,

$$X^2 + 2Y^2 + 5Z^2 = 40n + 32 \text{ with } X \equiv Y \pmod 5$$

has a solution. We will show that

$$X^2 + 2Y^2 + 5Z^2 = 40n + 32$$

has an integer solution X, Y and Z such that $X \equiv Y \pmod 5$ and $\gcd(XYZ, 2) = 1$. Assume that

$$X^2 + 2Y^2 + 5Z^2 = 40n + 32$$

Suppose that $(X, Y, Z) = (x, y, z)$ is an integer solution of $f(X, Y, Z) = 40n + 32$ such that $x \equiv y \pmod 5$ and $x \equiv y \equiv z \equiv 0 \pmod 2$. Then $x \equiv z \pmod 4$ and there is another representation

$$f(x, y, z) = f\left(\frac{-6x + 4y + 10z}{8}, \frac{2x - 4y + 10z}{8}, \frac{-2x - 4y - 2z}{8}\right).$$

satisfying

$$\frac{-3x+2y+5z}{4} - \frac{x-2y+5z}{4} = -x+y \equiv 0 \pmod 5.$$

Let

$$\tau_1 = \frac{1}{4}\begin{pmatrix} -3 & 2 & 5 \\ 1 & -2 & 5 \\ -1 & -2 & -1 \end{pmatrix}.$$

We know that τ_1 has an infinite order and there are only finitely many representations. This means that there exist a positive integer m such that each component of the vector $\tau_1^m(x,y,z) = (x_m, y_m, z_m)$ is odd and $x_m \equiv y_m \pmod 5$, if (x,y,z) is not an eigenvector of τ_1. Since the eigenspace of τ_1 is $z_1 = (-2t, t, 0)$ and $f(z_1) = 6t^2$. It is not the form of $40n+32$. The quadratic form $X^2+2Y^2+5Z^2 = 40n+32$ has an integer solution X, Y and Z such that $X \equiv Y \pmod 5$ and $\gcd(XYZ, 2) = 1$. Therefore, the sum $P_3(x) + P_7(y) + P_7(z)$ is universal over $\mathbb{Z}$.

(15) $P_3(x) + P_4(y) + P_9(z)$

By 3.1.2 and 3.1.3, the ternary sum $P_3(x) + P_4(y) + P_9(z)$ is universal over $\mathbb{Z}$ if and only if for any nonnegative integer n, the diophantine equation

$$X^2 + 7Y^2 + 56Z^2 = 56n + 32$$

has an integer solution X, Y and Z with $\gcd(XY, 2) = 1$. Let

$$M = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} \perp \langle 28 \rangle \quad \text{and} \quad N = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 8 & 1 \\ 2 & 1 & 8 \end{pmatrix}$$

Note that

$$\text{gen}(M) = \{[M], [N]\}$$

and $28n + 16$ is an eligible integer for $\text{gen}(M)$ for any nonnegative integer n. Furthermore, one may easily check that

$$N \prec_{4,0} M.$$

Hence M represents every positive integer of the form $28n+16$. Since

$$2f_M(X,Y,Z) = 4X^2 - 4XY + 8Y^2 + 56Z^2 = (Y-2X)^2 + 7Y^2 + 56Z^2,$$

the diophantine equation

$$X^2 + 7Y^2 + 56Z^2 = 56n + 32$$

has an integer solution X, Y and Z with $X \equiv Y \pmod 2$. Note that

$$f_M(X,Y,Z) = f_M\left(\frac{-X-2Y}{2}\frac{X-2Y}{2}, Z\right).$$

Since the order of

$$\tau = \begin{pmatrix} \frac{-1}{2} & -1 & 0 \\ \frac{1}{2} & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

has an infinite order, $f_M(X,Y,Z) = 28n+16$ has an integer solution with $X \equiv 1 \pmod 2$. Note that the eigenvector of τ corresponding to the eigenvalue 1 is $(0,0,t)$ and $f_M(0,0,t) = 28t^2 \neq 28n+16$ for any integer n. Therefore the sum $P_3(x) + P_4(y) + P_9(z)$ is universal over $\mathbb{Z}$.

This completes the proof $\square$

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국문초록

정수 x와 3 이상의 정수 m에 대하여 $p_m(x) = \frac{(m-2)x^2-(m-4)x}{2}$ 로 표현되는 정수를 위수 m의 일반화된 다각수라고 한다. 양의 정수 a, b, c와 3 이상의 정수 $i, j.k$에 대하여 방정식 $\Phi_{i,j,k}^{a,b,c}(x, y, z) = ap_i(x) + bp_j(y) + cp_k(z) = n$이 정수해 x, y, z를 가지면 일반화된 삼변수 다각수의 합 $\Phi_{i,j,k}^{a,b,c}(x, y, z) = ap_i(x) + bp_j(y) + cp_k(z)$을 보편합이라고 한다. 이 논문에서 우리는 (1.0.2)에서 나열된 15개의 일반화된 삼변수 다각수의 보편합을 증명한다. 이것은 Z.-W. Sun의 추측이다.

주요어휘: 일반화된 다각수, 보편 다각수 합, 삼변수 다각수 합, 삼변수 이차 형식의 표현

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